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Solutions to a stationary nonlinear Black–Scholes type equation

P. Amster, C.G. Averbuj, and M.C. Mariani *

*Dpto. de Matemática, Fac. de Cs. Exactas y Naturales, UBA Pab. I, Ciudad Universitaria (1428),
Capital, Argentina*

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Abstract

We study by topological methods a nonlinear differential equation generalizing the Black–Scholes formula for an option pricing model with stochastic volatility. We prove the existence of at least a solution of the stationary Dirichlet problem applying an upper and lower solutions method. Moreover, we construct a solution by an iterative procedure. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

In this work we study a nonlinear differential equation arising in an option pricing model. From the Black–Scholes model, if volatility is stochastic, the following PDE on the variables σ and S is obtained [3]:

$$\mathcal{L}f - \frac{1}{2}\rho\sigma^2 V f_{\sigma} = rf - rSf_S, \quad (1)$$

where \mathcal{L} is the operator given by

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}V^2 \sigma^2 \frac{\partial^2}{\partial \sigma^2} + \rho\sigma^2 S V \frac{\partial^2}{\partial S \partial \sigma}.$$

* Corresponding author.

E-mail addresses: pamster@dm.uba.ar (P. Amster), caverbuj@dm.uba.ar (C.G. Averbuj), mcmarian@dm.uba.ar (M.C. Mariani).

We study the stationary case for a generalization of (1) under Dirichlet conditions.

Applying a Newton-type iteration [2] we prove under appropriate conditions the existence of a solution in the Sobolev space $H^2(\Omega)$ which is obtained recursively. In Section 4 we obtain a solution under different assumptions, applying an upper and lower solutions method.

Our main interest is a better understanding of Black–Scholes type equations.

2. Black–Scholes type differential equations

The Black–Scholes equation for pricing options has been studied by many authors (see, for example, [1,4,6,8,9]).

In particular, stochastic volatility models are proposed: specifically, we shall consider as in [3] the following processes

$$dS_t = S_t \sigma_t dZ_t + S_t \mu dt,$$

$$d\sigma_t = V \sigma_t dW_t + \alpha \sigma_t dt,$$

where Z_t and W_t are two standard Brownian motions with correlation coefficient ρ , formally $E(dZ_t, dW_t) = \rho dt$. If $f(S, \sigma, t)$ is the price of an option depending on the price of the asset S , then by Ito's lemma [7], it holds

$$df(S, \sigma, t) = f_S dS + f_\sigma d\sigma + \mathcal{L}f dt.$$

Under an appropriate choice of the portfolio the stochastic term of the equation vanishes (for details, see [3]).

3. Stationary solutions to a nonlinear Black–Scholes type equation

We study the following stationary Dirichlet problem:

$$(1a) \quad \begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho \sigma^2 V S \frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho \sigma^2 V \frac{\partial f}{\partial \sigma} \\ = rg(f)f - rS \frac{\partial f}{\partial S} & \text{in } \Omega_0, \\ f = h_0 & \text{on } \partial\Omega_0, \end{cases}$$

with $g \in C^2(\mathbb{R})$, $h_0 \in H^2(\Omega_0)$, $\overline{\Omega}_0 \subset (0, a) \times (0, b)$ with $C^{1,1}$ boundary.

In this section we shall apply an iterative method in order to solve (1a).

Let us introduce the change of variables Φ given by $y = \log S$, $x = \sigma/V$; adding a parameter $\lambda \in [0, 1]$ into (1a) we obtain the following problem for $u(x, y) = f(S, \sigma)$ in the domain $\Omega = \Phi(\Omega_0)$:

$$(1b)_\lambda \quad \begin{cases} \Delta u + 2\rho \frac{\partial^2 u}{\partial x \partial y} = \lambda \left(\rho \frac{\partial u}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial u}{\partial y} + \frac{2rg(u)}{x^2 V^2} u \right) & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

For simplicity, we define

$$F\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \rho \frac{\partial u}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial u}{\partial y} + \frac{2r g(u)}{x^2 V^2} u$$

and the linear operator

$$Lu = \Delta u + 2\rho \frac{\partial^2 u}{\partial x \partial y}.$$

We remark that L is strictly elliptic for $\rho < 1$.

We start at a solution u_0 of $(1b)_{\lambda_0}$ and construct recursively a solution of $(1b)_{\lambda_0 + \varepsilon}$ for some step ε . Thus, we have solutions for $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$, and if ε can be chosen uniformly the procedure gives a solution of problem $(1b)_1$.

In order to define a convergent sequence we apply Newton's method: let $\psi : H^2(\Omega) \rightarrow L^2(\Omega)$ be given by

$$\psi(u) = Lu - (\lambda_0 + \varepsilon)F(x, u, \nabla u)$$

and define

$$u_{n+1} = u_n - [D\psi(u_n)]^{-1}(\psi(u_n)).$$

Under appropriate conditions the differential $D\psi(u)$ given by

$$\begin{aligned} D\psi(u)(\varphi) &= L\varphi - (\lambda_0 + \varepsilon) \left(\frac{\partial F}{\partial u}(x, u, \nabla u)\varphi + \frac{\partial F}{\partial u_x}(x, u, \nabla u) \frac{\partial \varphi}{\partial x} \right. \\ &\quad \left. + \frac{\partial F}{\partial u_y}(x, u, \nabla u) \frac{\partial \varphi}{\partial y} \right) \\ &= L\varphi - (\lambda_0 + \varepsilon) \left(\frac{2r[g(u)u]'}{x^2 V^2} \varphi + \rho \frac{\partial \varphi}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial \varphi}{\partial y} \right) \end{aligned}$$

is invertible. Hence, the sequence is well defined and converges quadratically to a zero of ψ .

We remark that if $z = [D\psi(u_n)]^{-1}(\psi(u_n))$ then

$$\begin{aligned} Lz - (\lambda_0 + \varepsilon) \left(\frac{2r[g(u_n)u_n]'}{x^2 V^2} z + \rho \frac{\partial z}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial z}{\partial y} \right) \\ = \psi(u_n) = Lu_n - F(x, u_n, \nabla u_n). \end{aligned}$$

Then $u_{n+1} = u_n - z$ is the unique solution of the linear problem

$$\begin{aligned} Lu_{n+1} &= (\lambda_0 + \varepsilon) \left[\rho \left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x} \right) + \left(1 - \frac{2r}{x^2 V^2}\right) \left(\frac{\partial u_{n+1}}{\partial y} - \frac{\partial u_n}{\partial y} \right) \right. \\ &\quad \left. + \frac{2r(g(u_n) + u_n g'(u_n))}{x^2 V^2} (u_{n+1} - u_n) + F\left(x, u_n, \frac{\partial u_n}{\partial x}, \frac{\partial u_n}{\partial y}\right) \right] \end{aligned}$$

with the boundary condition

$$u_{n+1} = h \quad \text{on } \partial\Omega.$$

If we assume that $g(u)u$ is nondecreasing with respect to u , then the following lemma shows that $\{u_n\}$ is well defined.

Lemma 1. *Let $s \in C(\overline{\Omega})$ and $L_s: H^2(\Omega) \rightarrow L^2(\Omega)$ be the linear operator given by*

$$L_s z = Lz - \lambda \left[\rho \frac{\partial z}{\partial x} + \left(1 - \frac{2r}{x^2 V^2} \right) \frac{\partial z}{\partial y} + sz \right]$$

with $s \geq 0$ and $0 \leq \lambda \leq 1$. Then $L_s|_{H_0^1(\Omega)}$ is invertible and onto. Moreover, there exists a constant $c > 0$ depending only on $\|s\|_\infty$ such that $\|z\|_{2,2} \leq c \|L_s z\|_2$ for any $z \in H^2 \cap H_0^1(\Omega)$.

Proof. By classical results [5], the linear problem

$$L_s z = \varphi \quad \text{in } \Omega, \quad z|_{\partial\Omega} = 0$$

is uniquely solvable in $H^2(\Omega)$ for any $\varphi \in L^2(\Omega)$. Assume the existence of $s_n \geq 0$ and $u_n \in H^2 \cap H_0^1(\Omega)$ such that

$$\|s_n\|_\infty \leq M, \quad \|u_n\|_{2,2} = 1, \quad \|L_{s_n} u_n\|_2 \rightarrow 0.$$

As $\int_\Omega L_{s_n} u_n \cdot u_n \rightarrow 0$, we obtain

$$-\int_\Omega L u_n \cdot u_n + \lambda \left[\rho \int_\Omega \frac{\partial u_n}{\partial x} u_n + \int_\Omega \left(1 - \frac{2r}{x^2 V^2} \right) \frac{\partial u_n}{\partial y} u_n + \int_\Omega s_n u_n^2 \right] \rightarrow 0.$$

By ellipticity, $-\int_\Omega L u_n \cdot u_n \geq k \|u_n\|_{1,2}^2$ for some positive constant k . Moreover, if we define the fields

$$F_1(x, y) = (u_n^2, 0), \quad F_2(x, y) = \left(1 - \frac{2r}{x^2 V^2} \right) (0, u_n^2),$$

we see that

$$\int_\Omega \frac{\partial u_n}{\partial x} u_n = \frac{1}{2} \int_\Omega \operatorname{div} F_1 = \frac{1}{2} \int_{\partial\Omega} F_1 \cdot \nu \, dS = 0$$

and

$$\int_\Omega \left(1 - \frac{2r}{x^2 V^2} \right) \frac{\partial u_n}{\partial y} u_n = \frac{1}{2} \int_\Omega \operatorname{div} F_2 = \frac{1}{2} \int_{\partial\Omega} F_2 \cdot \nu \, dS = 0.$$

As $s_n \geq 0$, we deduce that $\|u_n\|_{1,2} \rightarrow 0$. This implies that $\|L u_n\|_2 \rightarrow 0$, which contradicts the invertibility of L . \square

Theorem 2. Let us assume that $g(u)u$ is nondecreasing with respect to u . Then there exists ε such that $\{u_n\}$ converges for the norm $\|\cdot\|_{2,2}$ to a solution of (1b) $_{\lambda_0+\varepsilon}$.

Proof. Let $z_n = u_{n+1} - u_n$, and consider

$$s_n(x, y) = \frac{2r(g(u_n) + u_n g'(u_n))}{x^2 V^2} \geq 0.$$

Then for $\lambda = \lambda_0 + \varepsilon$ it holds

$$\begin{aligned} L_{s_n} z_n &= (\lambda_0 + \varepsilon) F(x, u_n, \nabla u_n) - L u_n \\ &= (\lambda_0 + \varepsilon) [F(x, u_n, \nabla u_n) - F(x, u_{n-1}, \nabla u_{n-1}) \\ &\quad - D F(x, u_{n-1}, \nabla u_{n-1})(x, z_{n-1}, \nabla z_{n-1})] \\ &= (\lambda_0 + \varepsilon) \frac{r}{x^2 V^2} (\xi_n g''(\xi_n) + 2g'(\xi_n)) z_{n-1}^2 \end{aligned}$$

for some mean value $\xi_n(x, y)$ between u_n and u_{n-1} . If $\|u_n - u_0\|_{2,2} \leq R$ for some constant R and any $n \leq N$, then there exists a constant K such that

$$\|g(u_n) + u_n g'(u_n)\|_\infty \leq K, \quad \|\xi_n g''(\xi_n) + 2g'(\xi_n)\|_\infty \leq K.$$

By the previous lemma, we have that

$$\|z_n\|_{2,2} \leq c \|L_{s_n} z_n\|_2 \leq c(\lambda_0 + \varepsilon) \frac{r}{x^2 V^2} K \|z_{n-1}^2\|_2 \leq c_1 \|z_{n-1}\|_{2,2}^2$$

for some constant c_1 . Inductively,

$$\|z_n\|_{2,2} \leq (c_1 \|z_0\|_{2,2})^{2^n - 1} \|z_0\|_{2,2}$$

and hence

$$\|u_{N+1} - u_0\|_{2,2} \leq \sum_{j=0}^N T^{2^j - 1} \|z_0\|_{2,2}$$

for $T = c_1 \|z_0\|_{2,2}$. As

$$L_{s_0} z_0 = \varepsilon F(x, u_0, \nabla u_0),$$

we may choose ε such that $\|z_0\|_{2,2} < 1/c_1$. Hence, $T < 1$ and

$$\|u_{N+1} - u_0\|_{2,2} \leq \varepsilon \|F(x, u_0, \nabla u_0)\|_2 \frac{1}{1 - T}$$

Thus, taking ε small we may assume that $\|u_n - u_0\|_{2,2} \leq R$ for any n , and the previous computations imply that $\{u_n\}$ is a Cauchy sequence. This completes the proof. \square

The following theorem shows that under an extra assumption the step ε may be chosen uniformly.

Theorem 3. *Let us assume that*

$$0 \leq \frac{d[g(u)u]}{du} \leq M$$

for some constant M . Then the step ε of Theorem 2 may be chosen independent of u_0 . Hence, there exists a sequence

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = 1,$$

where the solutions u_j of $(1b)_{\lambda_j}$ are constructed as in Theorem 2, and u_N is the unique solution of the original problem.

Proof. It suffices to prove the existence of a constant C such that if u satisfies $(1b)_\lambda$ for some λ , then $\|u\|_{2,2} \leq C$: Indeed, in that case in the proof of Theorem 2 we have that $\|u_0\|_{2,2} \leq C$, and by the imbedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ we conclude that the constant K can be considered such that

$$\|g(u) + ug'(u)\|_\infty \leq K, \quad \|ug''(u) + 2g'(u)\|_\infty \leq K$$

for any $u \in B_{C+R}(u_0) \subset H^2(\Omega)$.

If u satisfies $(1b)_\lambda$, with the notation of Lemma 1 there exists a mean value ξ such that $L_s u = 0$ for $s = 2r^2/x^2 V^2(\xi g'(\xi) + g(\xi))$, with $\|s\|_\infty \leq M$. As

$$\|u - h\|_{2,2} \leq c \|L_s h\|_{2,2} \leq C$$

for some constant C independent of u , our claim is proved. \square

4. An upper and lower solutions method for (1a)

In this section we obtain solutions of (1a) by an upper and lower solutions method. As before, we shall consider the equivalent problem $(1b)_1$. Our main result is the following:

Theorem 4. *Let us assume that there exists a nonnegative constant α such that:*

- (i) $h(x, y) \leq \alpha$ for $(x, y) \in \partial\Omega$,
- (ii) $g(\alpha) \geq 0$.

Then the problem $(1b)_1$ admits a solution $u \in H^2(\Omega)$ with $0 \leq u \leq \alpha$.

Proof. Let

$$M = \sup_{0 \leq u \leq \alpha} g(u) + ug'(u),$$

and choose a positive constant s satisfying

$$s > \frac{2r}{x^2 V^2} M$$

for any x such that $(x, y) \in \overline{\Omega}$ for some y . Hence the function

$$\psi(x, u) := \frac{2rg(u)}{x^2 V^2} u - su$$

is strictly decreasing with respect to u for $u \in [0, \alpha]$. We define a sequence $\{u_n\}$ in the following way: set $u_0 \equiv \alpha$, and consider u_{n+1} as the unique solution (given by Lemma 1) of the linear problem

$$\begin{cases} Lu - su = \frac{2rg(u_n)}{x^2 V^2} u_n - su_n & \text{in } \Omega, \\ u|_{\partial\Omega} = h, \end{cases}$$

where

$$Lu = \Delta u + 2\rho \frac{\partial u^2}{\partial x \partial y} - \rho \frac{\partial u}{\partial x} - \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial u}{\partial y}.$$

We claim that:

- (i) $0 \leq u_n \leq \alpha$ for every n .
- (ii) For any $(x, y) \in \Omega$ the sequence $\{u_n(x, y)\}$ is nonincreasing.

In order to prove claims (i) and (ii) we proceed by induction: assume, for example, that $u_1(x_0, y_0) > \alpha$ for some $(x_0, y_0) \in \overline{\Omega}$. As $u_1|_{\partial\Omega} = h \leq \alpha$, we deduce that $(x_0, y_0) \in \Omega$ and we may assume that (x_0, y_0) is a maximum. As $\nabla u_1(x_0, y_0) = 0$, we have that

$$\left(\Delta u_1 + 2\rho \frac{\partial^2 u_1}{\partial x \partial y} - su_1 \right) \Big|_{(x_0, y_0)} = \frac{2rg(\alpha)}{x^2 V^2} \alpha - s\alpha \geq -s\alpha.$$

Hence

$$\left(\Delta u_1 + 2\rho \frac{\partial^2 u_1}{\partial x \partial y} \right) \Big|_{(x_0, y_0)} = s[u_1(x_0, y_0) - \alpha] > 0,$$

which contradicts the maximum principle. On the other hand, as ψ is nonincreasing we have that $Lu_1 - su_1 = \psi(x, \alpha) \leq \psi(x, 0) \leq 0$, and being $u_1|_{\partial\Omega} = h \geq 0$ we obtain by the minimum principle that $u_1 \geq 0$.

Next, we assume as inductive hypothesis that $0 \leq u_n \leq u_{n-1}$. As before, if $[u_{n+1} - u_n](x_0, y_0) > 0$ is maximum, then

$$\begin{aligned} (Lu_{n+1} - su_{n+1})|_{(x_0, y_0)} &= \left(\frac{2rg(u_n)}{x^2 V^2} u_n - su_n \right) \Big|_{(x_0, y_0)} \\ &\geq \left(\frac{2rg(u_{n-1})}{x^2 V^2} u_{n-1} - su_{n-1} \right) \Big|_{(x_0, y_0)} = (Lu_n - su_n)|_{(x_0, y_0)}. \end{aligned}$$

Hence, as $\nabla u_{n+1}(x_0, y_0) = \nabla u_n(x_0, y_0)$, we conclude that

$$\left(\Delta(u_{n+1} - u_n) + 2\rho \frac{\partial^2(u_{n+1} - u_n)}{\partial x \partial y} \right) \Big|_{(x_0, y_0)} \\ = s[u_{n+1}(x_0, y_0) - u_n(x_0, y_0)] > 0,$$

a contradiction. The inequality $u_{n+1} \geq 0$ follows in the same way as before.

Hence, there exists a function $u: \overline{\Omega} \rightarrow \mathbb{R}$ such that $u_n(x, y) \rightarrow u(x, y)$ for every (x, y) . By Lemma 1, there exists $H \in H^2(\Omega)$ such that

$$LH - sH = 0, \quad H|_{\partial\Omega} = h.$$

Moreover,

$$\|u_{n+1} - H\|_{2,2} \leq c \|Lu_{n+1} - su_{n+1} - (LH - sH)\|_2 = c \|\psi(\cdot, u_n)\|_2$$

and $|\psi(\cdot, u_n)| \leq K$ for a constant K independent of n . Hence, the sequence $\{u_n\}$ is bounded in $H^2(\Omega)$. Fix p such that $2 < p < \infty$, and suppose that $u_n \not\rightarrow u$ in $W^{1,p}(\Omega)$. Then there exists a subsequence $\{u_{n_j}\}$ with

$$\|u_{n_j} - u\|_{1,p} \geq \varepsilon$$

for some $\varepsilon > 0$. By the compactness of the imbedding $H^2(\Omega) \hookrightarrow W^{1,p}(\Omega)$ the sequence $\{u_{n_j}\}$ admits a subsequence that converges in $W^{1,p}(\Omega)$ to some v with $\|v - u\|_{1,p} \geq \varepsilon$, a contradiction since $u_n \rightarrow u$ pointwise. Thus, taking limit in the equality

$$Lu_{n+1} - su_{n+1} = \frac{2rg(u_n)}{x^2 V^2} u_n - su_n,$$

we easily conclude that

$$Lu = \frac{2rg(u)}{x^2 V^2} u$$

and the proof is complete. \square

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